ACM 30020 Advanced Mathematical Methods

Sturm Liouville Problem (SLP)

SL equation

A classical "'Sturm-Liouville equation"', is a real second-order linear differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+q(x)y=\lambda r(x)y, \tag{1}$$

In the simplest of cases all coefficients are continuous on the finite closed interval $[a,b]$, and $p(x)$ has continuous derivative. In this case y is called a "solution" if it is continuously differentiable on (a,b) and satisfies the equation (1) at every point in (a,b). In addition, the unknown function y is required to satisfy boundary conditions. The function $r(x)$, is called the "weight" or "density" function.

The number of "famous" differential equations could be represented in the SL form:

- Bessel's equation

  $$x^2y''+xy'+(x^2-\nu^2)y=0$$

  can be written in Sturm-Liouville form as

  $$(xy')'+(x-\nu^2/x)y=0.$$

- The Legendre equation:

  $$(1-x^2)y''-2xy'+\nu(\nu+1)y=0$$

  can easily be put into SL form, since $(1-x^2)'=-2x$, so, the Legendre equation is equivalent to

  $$[(1-x^2)y']'+\nu(\nu+1)y=0$$

The general way to convert the 2nd order linear ODE to the SL form is to use an *integrating factor* $\mu(x)$ such that the equation

$$P(x)y''+Q(x)y'+R(x)y=0$$

multiplied by $\mu(x)$ would have the SP form. One cane easily show that

$$\mu(x)=\frac{1}{P(x)}\exp(\int Q(x)/P(x)dx)$$

does the job.

SL Boundary Value Problem (SL-BVP)

We introduce the SL-operator as

$$L[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y$$

and consider the SL equation

$$L[y] + \lambda r(x)y = 0 \ , \tag{2}$$

where $p > 0$, $r \geq 0$ and p, q and r are continuous functions on the interval $[a, b]$; along the with BC

$$B_a[y] = \alpha_1 y(a) + \alpha_2 p(a)y'(a) = 0 \qquad\qquad B_b[y] = \beta_1 y(b) + \beta_2 p(b)y'(b) = 0 \ , \tag{3}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$

The problem of finding a complex number $\lambda = \mu$ such that the BVP (2)-(3) *has a non-trivial solution is called SLP. The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(:, \mu)$ is called an eigenfunction*

There are three types of SLP:

1. A SLP is called *regular* if $p > 0$, and $r > 0$ on $[a, b]$
2. A SLP is called *singular* if $p > 0$ on (a, b), $r \geq 0$ on $[a, b]$ and $p(a) = p(b) = 0$.
3. A SLP is called *periodic* if $p > 0$, $r > 0$ and p, q and r are continuous functions on $[a, b]$; along with the following BC:

$$y(a) = y(b) \qquad\qquad y'(a) = y'(b) \ .$$

The most common types of SLP are regular and periodic, which will be discussed in more detail.

Example 1

For $\lambda \in R$ solve

$$y'' + \lambda y = 0, \ \ y(0) = y'(\pi) = 0$$

We consider three cases corresponding to values of λ:

- $\lambda = -\mu^2 < 0$

  The general solution of the ODE is given as

  $$y = Ae^{-\mu x} + Be^{\mu x}$$

  By substituting BC we obtain the following system:

  $$\begin{cases} A + B & = 0 \\ -Ae^{-\mu\pi} + Be^{\mu\pi} & = 0 \end{cases}$$

  This system has only trivial solution $A = B = 0$ (determinant of its matrix of coefficients is different from 0)

- $\lambda = 0$

  In this case the problem has a solution $y = Ax + B$ and by substituting BC one can check that $A = B = 0$ (we get a trivial solution as well).

- $\lambda = \mu^2 > 0$

  The general solution of the ODE is given as

$$y = A\cos(\mu x) + B\sin(\mu x)$$

  By substituting BC we obtain the following system:

$$\begin{cases} A = 0 \\ B\cos(\mu\pi) = 0 \end{cases}$$

  This problem has non-trivial solution (enabling $B \neq 0$) only when $\cos(\mu\pi) = 0$ or $\mu = (2n-1)/2$. Therefore the eigenvalues λ_n could be written as

$$\lambda_n = \frac{(2n-1)^2}{4}, \;\; n = 0, \pm1, \pm2, \ldots$$

  and the eigenfunctions (we choose $B_n = 1$) are

$$\varphi_n = \sin(\sqrt{\lambda_n}x)$$

  Note that all the eigenvalues λ_n are positive and the eigenfunctions corresponding to each eigenvalue form a one dimensional vector space, and so the eigenfunctions are unique up to a constant multiple

Example 2

For $\lambda \in R$ solve

$$y'' + \lambda y = 0, \;\; y(0) - y(\pi) = 0, \;\; y'(0) - y'(\pi) = 0$$

These BC are called "periodic BC".

We consider three cases corresponding to values of λ:

- $\lambda = -\mu^2 < 0$

  The general solution of the ODE is given as

$$y = Ae^{-\mu x} + Be^{\mu x}$$

  By substituting BC we obtain the following system:

$$\begin{cases} A(1 - e^{-\mu\pi}) + B(1 - e^{\mu\pi}) & = 0 \\ A(-1 + e^{-\mu\pi}) + B(1 - e^{\mu\pi}) & = 0 \end{cases}$$

  This system has only trivial solution $A = B = 0$ (for $\mu \neq 0$)

- $\lambda = 0$

  In this case the problem has a solution $y = Ax + B$ and by substituting BC we obtain $A = 0$ and B is an arbitrary constant. This corresponds to the eigenvalue $\lambda_0 = 0$ and the eigenfunction $\varphi_0 = 1$ (we set $B = 1$).

  Note that this eigenvalue is simple. The eigenvalue is called simple, if its eigenspace is of dimension one; otherwise the eigenvalue is called multiple.

- $\lambda = \mu^2 > 0$

  The general solution of the ODE is given as

  $$y = A\cos(\mu x) + B\sin(\mu x)$$

  By substituting BC we obtain the following system:

  $$\begin{cases} A(1-\cos(\mu\pi)) - B\sin(\mu\pi) & = 0 \\ A\sin(\mu\pi) + B(1-\cos(\mu\pi)) & = 0 \end{cases}$$

  This problem has a non-trivial solution only when the determinant of the matrix of coefficients $D(\mu) = 2 - \cos(\mu\pi) = 0$. This corresponds to $\mu = 2n, \;\; n = \pm1, \pm2, ...$ and hence $\lambda_n = 4n^2$.

  The eigenfunctions corresponding to λ_n are given by ($A = B = 1$)

  $$\varphi_n = \cos(\sqrt{\lambda_n}x) \qquad\qquad \psi_n = \sin(\sqrt{\lambda_n}x)$$

  Note that all the eigenvalues λ_n are positive and there are two linearly independent eigenfunctions corresponding to each eigenvalue, so they are not unique.

Regular SLP

Properties:

1. *The eigenvalues of the regular SLP are real*

   PROOF: Suppose $\lambda \in C$ is an eigenvalue of the regular SLP and let y be corresponding eigenfunction. That is,

   $$L[y] + \lambda r(x)y = 0, \;\; \alpha_1 y(a) + \alpha_2 p(a)y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 p(b)y'(b) = 0 \;, \tag{4}$$

   Taking the complex conjugates we get

   $$L[\bar{y}] + \bar{\lambda} r(x)\bar{y} = 0, \;\; \alpha_1 \bar{y}(a) + \alpha_2 p(a)\bar{y}'(a) = 0, \qquad \beta_1 \bar{y}(b) + \beta_2 p(b)\bar{y}'(b) = 0 \;, \tag{5}$$

   Multiplying the ODE in (4) with $\bar{y}$ and the ODE in (5) with y and subtracting one from another yields

   $$[p(y'\bar{y} - y\bar{y}')]' + (\lambda - \bar{\lambda})ry\bar{y} = 0$$

   Integrating the last expression we obtain

   $$[p(y'\bar{y} - y\bar{y}')]|_a^b = -(\lambda - \bar{\lambda})\int_a^b r|y|^2 dx = 0$$

   The LHS of the last identity is zero due to the BC. Thus we have

   $$(\lambda - \bar{\lambda})\int_a^b r|y|^2 dx = 0$$

   From the definition of regular SLP we know that $r > 0$ and y as an eigenfunction is different from zero as well. Therefore the only way to satisfy the identity is to set $\lambda = \bar{\lambda}$, which means that λ is real.

2. The eigenfunctions of a regular SLP corresponding to the distinct eigenvalues are orthogonal w.r.t. the weight function $r(x)$ on $[a, b]$. By other words, if the eigenfunctions u and v correspond to the distinct eigenvalues λ and μ then

$$\int_a^b r(x)u(x)v(x)dx = 0$$

PROOF: As in the previous case we write the SL equations for functions u and v, multiply one for u by v and vice versa and subtract one equation from another. As result we get

$$[p(u'v - v'u)]' + (\lambda - \mu)ruv = 0$$

Integrating the last expression we obtain

$$[p(u'v - uv')]|_a^b = -(\lambda - \mu)\int_a^b ruvdx = 0$$

The LHS of the last identity is zero due to the BC. Thus we have

$$(\lambda - \mu)\int_a^b ruvdx = 0$$

We know that $\lambda \neq \mu$ therefore $\int_a^b ruvdx = 0$, which confirms orthogonality.

3. The eigenvalues of the regular SLP are simple. Thus an eigenfunction that corresponds to an eigenvalue is unique up to a constant multiple.

4. The regular SL operator L is self-adjoint: if whenever $u, v \in C^2(a, b) \cap C^1[a, b]$ and satisfy the regular SLP and BC then

$$\int_a^b (vL[u] - uL[v])dx = 0$$

PROOF:

$$\int_a^b vL[u]dx = \int_a^b [-v(pu')] + vqu]dx = \text{(integration by parts)}$$
$$- pu'v|_a^b + \int_a^b (pu'v' + quv)dx$$

By symmetry we see that

$$\int_a^b uL[v]dx = \int_a^b [-u(pv')] + uqv]dx = \text{(integration by parts)}$$
$$- puv'|_a^b + \int_a^b (pu'v' + quv)dx$$

Subtracting one expression from another and applying BC $B_a[u] = B_a[v] = B_b[u] = B_b[v] = 0$ we obtain:

$$\int_a^b (vL[u] - uL[v])dx = [p(u'v - uv')]|_a^b = 0\ .$$

Theorem: A self-adjoint regular SLP has an infinite number of real eigenvalues λ_n that are simple and satisfying

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ...$$

with $\lim_{n\to\infty} \lambda_n = \infty$

Periodic SLP

Properties:

1. *The eigenvalues (if any) of the periodic SLP are real*
2. The eigenfunctions of a periodic SLP corresponding to the distinct eigenvalues are orthogonal w.r.t. the weight function $r(x)$ on $[a, b]$. By other words, if the eigenfunctions u and v correspond to the distinct eigenvalues λ and μ then

$$\int_a^b r(x)u(x)v(x)dx = 0$$

3. The eigenvalues of the regular SLP are *not simple*. Thus an eigenfunction that corresponds to an eigenvalue is not unique.
4. The periodic SL operator L is self-adjoint

All the proofs for periodic SLP are similar to the regular SLP and rely on the BC.

Theorem: A self-adjoint periodic SLP has an infinite number of real eigenvalues λ_n satisfying

$$-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n \leq ...$$

The first eigenvalue λ_1 is simple. The number of linearly independent eigenfunctions corresponding to any eigenvalue $\lambda_n = \mu$ ($n > 1$) is equal to the number of times μ is repeated in the above listing.

Representation of solutions and numerical calculation

The SL equation with boundary conditions may be solved in practice by a variety of numerical methods. In difficult cases, one may need to carry out the intermediate calculations to several hundred decimal places of accuracy in order to obtain the eigenvalues correctly to a few decimal places.

There are the most common methods used for this purpose:

- *Shooting methods.* These methods proceed by guessing a value of λ, solving an initial value problem defined by the boundary conditions at one endpoint, say, a, of the interval $[a, b]$, comparing the value this solution takes at the other endpoint b with the other desired boundary condition, and finally increasing or decreasing λ as necessary to correct the original value. This strategy is not applicable for locating complex eigenvalues.
- *Finite difference method.* The method involves approximation of SL on the small sub-intervals of $[a, b]$ using the first few terms of Taylor series expansion of y.

- The Spectral Parameter Power Series (SPPS) method. It makes use of a generalization of the following fact about second order ordinary differential equations: if y is a solution which does not vanish at any point of $[a, b]$, then the function

$$y(x)\int_a^x \frac{\mathrm{d}t}{p(t)y(t)^2}$$

is a solution of the same equation and is linearly independent from y. Further, all solutions are linear combinations of these two solutions. In the SPPS algorithm, one must begin with an arbitrary value λ_0^* (often $\lambda_0^* = 0$; it does not need to be an eigenvalue) and any solution y_0 of SLP with $\lambda = \lambda_0^*$ which does not vanish on $[a, b]$.

Two sequences of functions X^n ans $\tilde{X}^n$ referred to as "iterated integrals", are defined recursively as follows. First when $n = 0$, they are taken to be identically equal to 1 on $[a, b]$. To obtain the next functions they are multiplied alternately by $1/(py_0^2)$ and ry_0^2 and integrated, specifically

$$\begin{aligned}
X^n(x) &= -\int_a^x X^{n-1}(t)p(t)^{-1}y_0(t)^{-2}dt, \ \ n \text{ is odd}\\
X^n(x) &= \int_a^x X^{n-1}(t)r(t)y_0(t)^{2}dt, \ \ n \text{ is even}\\
\tilde{X}^n(x) &= -\int_a^x \tilde{X}^{n-1}(t)p(t)^{-1}y_0(t)^{-2}dt, \ \ n \text{ is even}\\
\tilde{X}^n(x) &= \int_a^x \tilde{X}^{n-1}(t)r(t)y_0(t)^{2}dt, \ \ n \text{ is odd}
\end{aligned}$$

when $n > 0$. The resulting iterated integrals are now applied as coefficients in the following two power series in λ:

$$u_0 = y_0\sum_{k=0}^{\infty}(\lambda-\lambda_0^*)^k\tilde{X}^{2k} \ \text{ and } \ u_1 = y_0\sum_{k=0}^{\infty}(\lambda-\lambda_0^*)^k X^{2k+1}.$$

Then for any λ (real or complex), u_0 and u_1 are linearly independent solutions of the corresponding SL equation. (The functions $p(x)$ and $q(x)$ take part in this construction through their influence on the choice of y_0)

Next one chooses coefficients c_0 and c_1 so the combination $y = c_0u_0+c_1u_1$ satisfies the first boundary condition $B_a[y]$. This is simple to do since $X^n(a) = \tilde{X}^n(a) = 0$. The values of $X^n(b)$ and $\tilde{X}^n(b)$ provide the values of $u_0(b)$ and $u_1(b)$ and the derivatives $u_0'(b)$ and $u_1'(b)$, so the second boundary condition $B_b[y]$ becomes an equation in a power series in λ. For numerical work one may truncate this series to a finite number of terms, producing a calculable polynomial in λ whose roots are approximations of the sought-after eigenvalues. The SPPS method can, itself, be used to find a starting solution y_0.

Example: application of SLP to solution of PDEs

For a linear second order in one spatial dimension and first order in time of the form:

$$\begin{aligned}
f(x)\frac{\partial^2 u}{\partial x^2} + g(x)\frac{\partial u}{\partial x} + h(x)u &= \frac{\partial u}{\partial t} + k(t)u\\
u(a,t) &= u(b,t) = 0\\
u(x,0) &= s(x)
\end{aligned}$$

Let us apply separation of variables, which in doing we must impose that:

$$u(x,t) = X(x)T(t)$$

Then above PDE may be written as:

$$\frac{\hat{L}X(x)}{X(x)} = \frac{\hat{M}T(t)}{T(t)}$$

where

$$\hat{L} = f(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(x)\frac{\mathrm{d}}{\mathrm{d}x} + h(x) \quad \text{and} \quad \hat{M} = \frac{\mathrm{d}}{\mathrm{d}t} + k(t)$$

Since, by definition, $\hat{L}$ and $X(x)$ are independent of time t and $\hat{M}$ and $T(t)$ are independent of position x, then both sides of the above equation must be equal to a constant that we call λ. In such a case:

$$\begin{aligned} \hat{L}X(x) &= \lambda X(x) \\ X(a) &= X(b) = 0 \\ \hat{M}T(t) &= \lambda T(t). \end{aligned}$$

The first of these equations must be solved as a SLP. Since there is no general analytic (exact) solution to SLP we can assume we already have the solution to this problem, that is, we have the eigenfunctions $X_n(x)$ and eigenvalues λ_n. The second of these equations can be analytically solved once the eigenvalues are known:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}T_n(t) &= (\lambda_n - k(t))T_n(t) \\ T_n(t) &= a_n e^{-\left(\lambda_n t - \int_0^t k(\tau)\mathrm{d}\tau\right)} \end{aligned}$$

and finally

$$u(x,t) = \sum_n a_n X_n(x) e^{-\left(\lambda_n t - \int_0^t k(\tau)\mathrm{d}\tau\right)}$$

where

$$a_n = \frac{(X_n(x), s(x))}{(X_n(x), X_n(x))}$$

We compute a_n as an*inner product* defined as

$$(y(x), z(x)) = \int_a^b y(x)z(x)w(x)\mathrm{d}x$$

where (for our case)

$$w(x) = \frac{e^{\int \frac{g(x)}{f(x)}\mathrm{d}x}}{f(x)}$$